Journal of Engineering Mathematics 21: 139–148 (1987)

Free-surface oscillations of fluid in closed basins

P. McIVER and S.R. SMITH
Department of Mathematics and Statistics, Brunel University, Uxbridge, Middlesex UB8 3PH, England

Received 8 July 1986; accepted 28 January 1987

Abstract. The mild-slope equation, well-known in wave refraction theory, is used to calculate the natural frequencies of oscillation of fluid in a basin. The method may be applied to canals of variable cross-section and to axisymmetric basins provided that every point in the fluid lies directly beneath the free surface. Comparison is made with previously known solutions and some new results are presented for axisymmetric geometries.

1. Introduction

Over recent years considerable effort has gone into developing numerical methods for determining the natural frequencies of oscillation of fluid in a closed basin. These methods have obtained a high degree of sophistication and are capable of analysing most shapes of basin and of giving very accurate results. The review article by Moiseev and Petrov [1] and the paper by Fox and Kuttler [2] contain many references to this body of work.

Though these methods are very accurate their application is not a trivial task. Before carrying out large-scale computations it may be desirable to employ simpler solution procedures to examine trends as parameters are varied. A small number of exact solutions of the linearised equations are known and most of these are described in Lamb [3]. There are some approximate analytical procedures, see for example Miles [4], but these are usually restricted in their application. In the present work a simple solution procedure is proposed utilising the mild-slope equation widely used for wave-refraction calculations. Though a numerical solution is required the problem reduces to one of standard Sturm-Liouville type for which efficient and easy to use computer packages are available. The method is attractive in that changes of geometry are very easily investigated (often requiring the alteration of only a single line of computer code). Here application is made to oscillations in canals of arbitrary cross-section (with two-dimensional motion as a special case) and in basins with a vertical axis of symmetry.

2. Formulation

Consider the free time-harmonic oscillations, with radian frequency ω, of fluid in a closed basin. Choose Cartesian coordinates (x, y, z) with the z-axis directed vertically upwards and the mean free surface in the plane $z = 0$. The depth of the fluid is allowed to vary so that the bottom of the container is at $z = -h(x, y)$. A model equation describing the propagation of waves in regions of varying depth is the "mild-slope" equation

$$\nabla \cdot (p \nabla \eta) + \omega^2 q \eta = 0 \tag{2.1}$$

where $\eta(x, y)$ is the free surface displacement,

$$p = gh \frac{\tanh kh}{kh} q, \tag{2.2}$$

and

$$q = \frac{1}{2}\left(1 + \frac{2kh}{\sinh 2kh}\right). \tag{2.3}$$

The local wavenumber k is a function of depth, and hence of position, through the dispersion relation

$$\omega^2 = gk \tanh kh. \tag{2.4}$$

A derivation of equation (2.1) is given by Mei [5]. The mild-slope equation is derived from the usual linearised equations of water waves under the assumption that changes in the water depth are small on the scale of the wavelength (i.e., $\nabla h \ll kh$). It is widely used in wave refraction calculations and has been applied to the free oscillations of infinite bodies of water (i.e., to edge or trapped waves) by Smith and Sprinks [6] and McIver and Evans [7]. In the present work, equation (2.1) will be applied to the free oscillations of a finite mass of water in a closed container.

Previous work has indicated that the mild-slope assumption does not severely restrict the range of applicability of the equation. This is confirmed here, with not unreasonable results obtained even when the depth tends to zero with a vertical tangent at the free surface. The nature of the derivation of equation (2.1) does impose one restriction on the geometry; every point in the fluid must lie vertically beneath the free surface.

3. Horizontal canals

Here the depth is taken to vary in the x-direction only and the bed intersects the free surface in the lines $x = \pm a$; $z = 0$. For standing waves periodic in the y-direction with wavenumber l, η may be expressed in the form

$$\eta(x, y) = \xi(x) \cos ly, \tag{3.1}$$

in which case equation (2.1) reduces to

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(p \frac{\mathrm{d}\xi}{\mathrm{d}x}\right) + (\omega^2 q - l^2 p)\, \xi = 0. \tag{3.2}$$

If there are vertical walls at $y = 0$, D the allowable modes of oscillation have $l = m\pi/D$, $m = 0, 1, 2, \ldots$. It is convenient to non-dimensionalise by taking

$$x = aX, \quad \xi = aY, \quad p = \omega^2 a^2 P \tag{3.3}$$

to give

$$\frac{\mathrm{d}}{\mathrm{d}X}\left(P\frac{\mathrm{d}Y}{\mathrm{d}X}\right) + ((ka)^2 - (la)^2)\,PY = 0. \tag{3.4}$$

Equation (3.4) is of standard Sturm-Liouville form. For fixed frequency, both P and ka are functions of X through the prescribed depth function $h/a = H$ and $\lambda = (la)^2$ is the eigenvalue. If the bed intersects the free surface vertically and with zero curvature the boundary conditions

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = 0, \quad X = \pm 1 \tag{3.5}$$

may be applied directly and the numerical solution is straightforward. However, for a non-vertical intersection or a vertical intersection with non-zero curvature that particular end-point is a singular point of the differential equation and direct application of the boundary condition is not possible. In these cases an appropriate asymptotic solution is sought valid close to the end-point, the role of the appropriate boundary condition is to determine which of the two possible forms for the asymptotic solution is required. Then, for numerical purposes, a modified boundary condition is applied by insisting that the solution of (3.4) takes this asymptotic form at a chosen 'boundary matching point' close to, but not coincident with, the end-point. There is some freedom of choice in both the order of the asymptotic solution and the position of the boundary matching point.

To derive asymptotic solutions of equations (3.4) expansions of P and $(ka)^2P(=q)$ are required in terms of a coordinate u local to the end-point (i.e., $u = X - 1$ or $u = X + 1$); these expansions are given in the Appendix. For a non-vertical intersection with the local depth given by

$$H = \alpha_1 u + \alpha_2 u^2 + \mathrm{O}(u^3) \tag{3.6}$$

(e.g., a part-filled canal of circular cross-section) there is a regular singular point and the asymptotic solutions may be found by the method of Frobenius (Bender and Orszag [8], Section 3.3). The non-singular solution is

$$Y = 1 - \frac{\nu}{\alpha_1}u + \frac{1}{4}\left(\frac{\nu^2}{\alpha_1^2} + \frac{2\nu\alpha_2}{\alpha_1^2} - \nu^2 + \lambda\right)u^2 + \mathrm{O}(u^3) \tag{3.7}$$

where $\nu = \omega^2 a/g$. At a vertical intersection of non-zero curvature with

$$H = u^{1/2}(\beta + \mathrm{O}(u)) \tag{3.8}$$

(e.g., a half-filled canal of circular cross-section) there is an irregular singular point of the differential equation. The asymptotic solutions are readily found by the method of dominant balance (Bender and Orszag [8], Section 3.4) and the solution satisfying (3.5) is

$$Y = 1 - \frac{2}{3}\frac{\nu}{\beta}u^{3/2} + \tfrac{1}{3}(\lambda - \tfrac{2}{3}\nu^2)u^2 + \mathrm{O}(u^{5/2}). \tag{3.9}$$

The coefficients in the local depth expansions (3.6) and (3.8) can be found by a trivial calculation for most geometries. For canals of symmetric cross-section the computational domain may be halved by imposing the further conditions

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = 0, \quad X = 0 \tag{3.10}$$

for symmetrical modes and

$$Y = 0, \quad X = 0 \tag{3.11}$$

for antisymmetric modes. For all of the calculations reported here the NAG (Numerical Algorithms Group, Oxford, UK) library eigenvalue-finding routine DO2KDF was used.

The eigenvalue problem defined by equation (3.4) together with the appropriate boundary conditions yields λ for a prescribed value of the frequency parameter v. In practice the geometry, and hence λ, is known and values of v are sought. The coefficients in equation (3.4) have a complicated dependence on v through the dispersion relation (2.4) and so the inverse problem cannot be formulated directly, though it may be treated by a relatively simple numerical procedure. Let $\lambda = \Lambda(v)$ represent the solution of the eigenvalue problem for given v. Suppose the values of v corresponding to $\lambda = \lambda_0$ are required, these are the zeros of

$$F(v) = \Lambda(v) - \lambda_0 \tag{3.12}$$

which are readily determined by numerical methods (e.g., by bisection). Note that two-dimensional oscillations correspond to $\lambda = 0$ and must be solved for in this way.

4. Axisymmetric basins

Define polar coordinates (r, θ) in the (x, y) plane, take the depth to be a function of r only (so that the geometry is vertically axisymmetric) and the free surface to be bounded by the circle $r = a$, $z = 0$. Solutions of equation (2.1) are sought in the form

$$\eta(r, \theta) = \xi(r) \cos m\theta, \quad m = 0, 1, 2, \ldots, \tag{4.1}$$

so that, with the non-dimensionalisation defined by equations (3.3) and setting $r = aR$, equation (2.1) becomes

$$\frac{\mathrm{d}}{\mathrm{d}R}\left(RP\frac{\mathrm{d}Y}{\mathrm{d}R}\right) + \left((ka)^2 - \frac{m^2}{R^2}\right) RPY = 0. \tag{4.2}$$

For non-zero m there is a regular singular point at $R = 0$ and therefore a boundary condition cannot be applied directly. The local non-singular solution is readily found to be

$$Y = R^m(1 + \mathrm{O}(R)) \tag{4.3}$$

which is to be imposed at a boundary matching point close to the origin.

For a singular intersection point the asymptotic solution valid near $u(=R-1)=0$ may be found as described in Section 3. For a depth given locally by equation (3.6) the solution is

$$Y = 1 - \frac{v}{\alpha_1}u + \frac{1}{4}\left(\frac{v}{\alpha_1} + \frac{v^2}{\alpha_1^2} + \frac{2v\alpha_2}{\alpha_1^2} - v^2 + \lambda\right)u^2 + O(u^3) \tag{4.4}$$

where now $\lambda = m^2$, while when the depth is given by equation (3.8) the solution is identical to equation (3.9), to the order given.

For axisymmetric basins it is not appropriate to fix v and determine λ as non-integer values of m are unphysical. Hence an inverse procedure is used, using a function similar to that defined in equation (3.12).

5. Comparison with known solutions

The simplest exact solution of the linearised equations describing free oscillations in a canal is for a rectangular cross-section. However, for any constant-depth geometry the mild-slope equation is exact. A solution for a canal of triangular cross-section intersecting the free surface at 45° is given by Lamb ([3], p. 447). This geometry provides quite a severe test for the mild-slope equation (for some parameter values the criterion $\nabla h \ll kh$ is violated, particularly near the intersections with the free surface) and in addition the end-points are regular singular points. Smith [9] has performed many calculations for this case in order to assess the accuracy of the mild-slope solution and to determine the best position for the boundary matching point. A comparison was also made of results found using the asymptotic solution (3.7) correct to $O(u^2)$ with results found using the simpler solution correct to $O(u)$. A small sample from Smith's computations is given in Table 1; note there are two modes of oscillation for $v = 2$. For $u = 0.001$ the two chosen forms of the asymptotic solution give virtually identical results over the whole frequency range. Similar comparisons for other geometries suggest that the eigenvalues of the mild-slope equation (rather than the exact values) may be obtained accurate to at least two decimal places by choosing $u = 0.001$ and the asymptotic solution correct to $O(u)$ for a regular singular point and to $O(u^{3/2})$ for an irregular singular point. Seeking improved accuracy beyond this is normally not justified in view of the approximate nature of the mild-slope equation.

Table 1. Canal of triangular cross-section. Variation of boundary matching points (u = 0.05, 0.01, 0.001) with asymptotic solution (3.7) correct to (a) $O(u)$, (*b*) $O(u^2)$

	Exact	1a					
		(a)			(b)		
v \ u		0.05	0.01	0.001	0.05	0.01	0.001
0.1	0.45481	0.45606	0.45546	0.45544	0.45544	0.45544	0.45544
0.5	1.09135	1.10099	1.09889	1.09880	1.09880	1.09880	1.09880
1.0	1.69660	1.72409	1.71859	1.71835	1.71831	1.71834	1.71834
2.0	2.92083	3.0067	2.9801	2.9789	2.9781	2.9787	2.9789
	2.70823	2.7993	2.7694	2.7680	2.7667	2.7679	2.7679

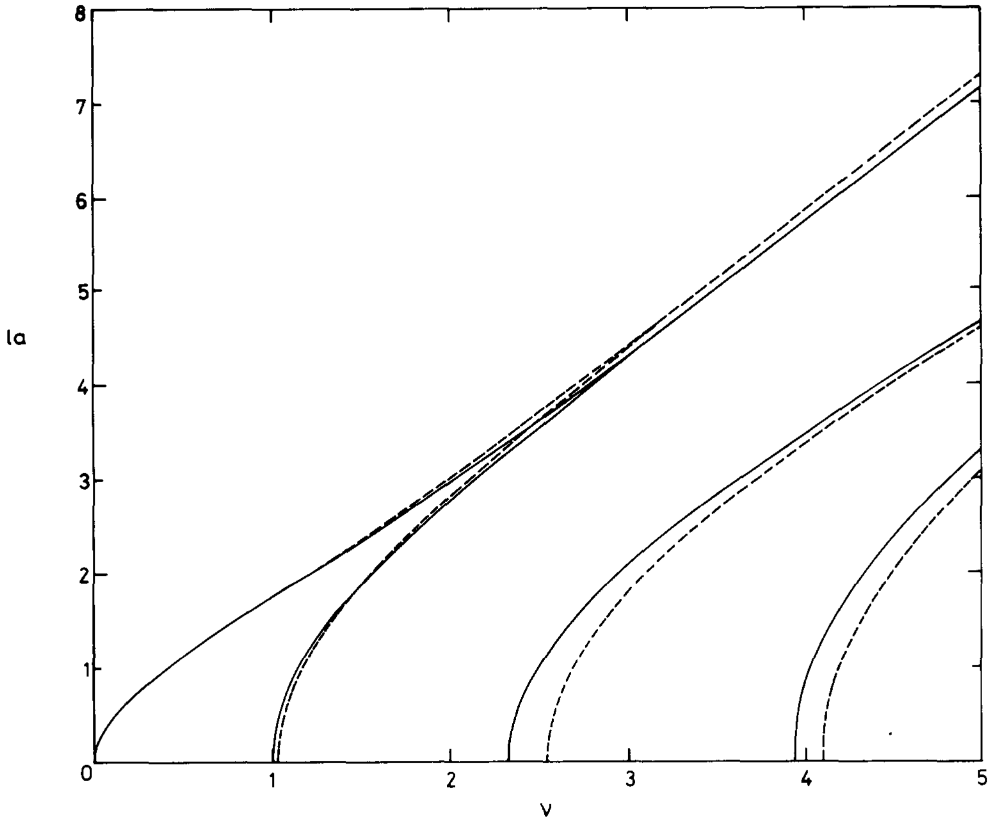

Fig. 1. The wavenumber la v. frequency parameter v for a canal of triangular cross-section; exact solution (——), mild-slope equation (– – –).

Table 2. Canal with circular cross-section of radius c. Variation of $\omega^2 c/g$ with fill-depth d for the lowest two-dimensional mode; (a) Kuttler and Sigillito [10], (b) mild-slope equation

d/c		0.4	0.6	0.8	1.0
$\omega^2 c/g$	(a)	1.10	1.16	1.25	1.36
	(b)	1.09	1.14	1.20	1.29

A graphical comparison of the exact and mild-slope solutions is made in Fig. 1 for the lowest four modes of the triangular canal. The best agreement is for the first two modes which are usually of most interest. Of particular interest is the lowest two-dimensional mode ($la = 0$) for which the exact solution is $v = 1$ and the mild-slope solution $v = 1.030$. The mild-slope assumption is most seriously violated in the regions local to the end-points because, for fixed v, as the depth goes to zero then so does kh. This error is most apparent at high frequencies when the fluid motion is negligible below a thin surface layer and the only influence of the bed shape is near the intersection points. From Fig. 1 it may be seen that this is offset to some extent for large la when the wave motion is along rather than across the slope.

A number of authors have made accurate numerical calculations for two-dimensional geometries. Kuttler and Sigillito [10] considered a circular cross-section filled to a variety of depths. A comparison of some of their results with those from the mild-slope equation is

Table 3. Canal with elliptical cross-section of axis ratio b/a (vertical/horizontal). Variation of $\omega^2 a/g$ with b/a for the lowest two-dimensional mode; (a) Chang and Wu [11], (b) mild-slope equation

b/a		0.50	0.75	1.25	1.75
$\omega^2 a/g$	(a)	0.95	1.20	1.44	1.49
	(b)	0.93	1.17	1.36	1.43

given in Table 2. Best agreement is for small fill-depths when the bed-slope is small everywhere. A comparison, in Table 3, with the calculations of Chang and Wu [11] for an elliptical cross-section shows a similar trend.

For axisymmetric geometries there is an additional complication as equation (4.2) has a regular singular point at the origin for non-zero m. There is an exact solution for an axisymmetric basin of constant depth (Mei [5] p. 188) and, as the mild-slope equation is exact for constant depth, the proper choice of boundary matching point may be assessed directly. With the simple asymptotic form (4.4) and the boundary matching point at $R = 0.001$, the first four modes for each value of m in the range zero to five (i.e., all of those tabulated by Mei) are given to at least four decimal places and these choices are used for all other calculations. Exact solutions of the linearised equations for axisymmetric geometries are rare. One other solution, possibly little known, is for a right-angled cone. In this case the potential

$$\phi = rz \cos \theta \cos \omega t \tag{5.1}$$

satisfies the full linearised water wave equations giving a frequency for the lowest mode of $v = 1$ which may be compared with $v = 1.017$ from the present work. Another axisymmetric solution is a numerical calculation by Budiansky [12] for the lowest mode in a half-filled spherical tank. He obtains $v = 1.57$ in comparison with $v = 1.49$ from the mild-slope equation.

6. Some new results

Though the previously described comparisons indicate that the mild-slope equation cannot compete with other solutions in terms of accuracy it does reproduce trends correctly as parameters are varied. It is in the examination of these trends, as for example fill depth or container shape varies, that approximate methods are valuable. Emphasis will now be placed on axisymmetric geometries as results for two-dimensional geometries are well-represented in the literature. The notation (m, j) is adopted to indicate the jth eigenfrequency (arranged in ascending order) for azimuthal wave number m.

In Table 4 results are compared for three different shapes of container; a cylindrical container with water depth equal to the radius, a half-filled sphere and a right-angled cone. With the chosen scalings the three containers each have effectively the same free-surface area and, therefore, contain very different volumes of fluid. For some applications it might be appropriate to consider containers of equal volume. For example, if the depth of the cylindrical container is adjusted to give the same volume of fluid as in a conical container of the same free-surface area, the frequency of the lowest mode (i.e., mode (1, 1)) is approximately the same for each container.

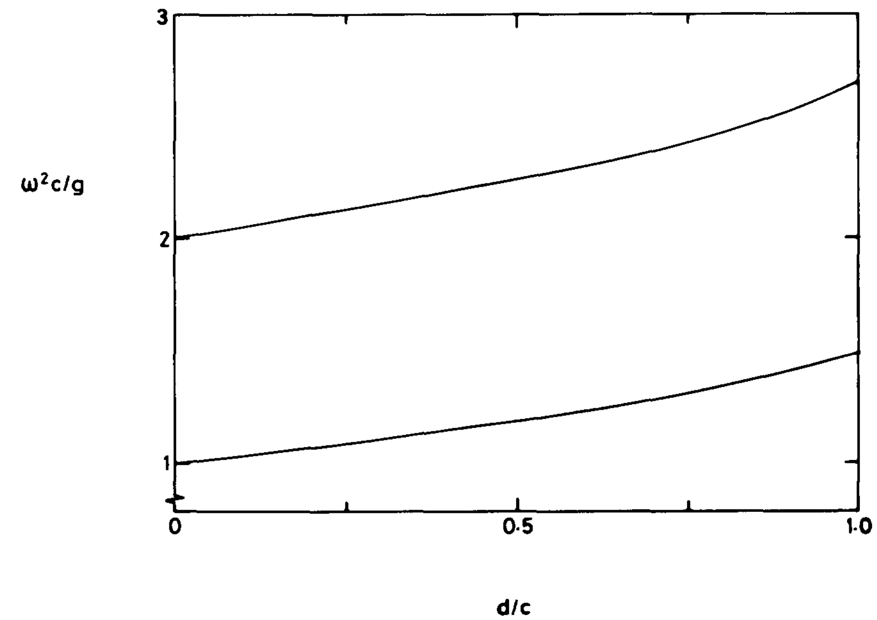

Fig. 2. The frequencies of the lowest two modes ((1, 1) and (2, 1)) for a spherical container of radius c filled to a depth d.

Table 4. Comparison of cylindrical (depth = radius), spherical and conical containers of radius a at free surface giving $\omega^2 a/g$ for a number of modes (N.B. the mode (0, 1) has $\omega = 0$)

Mode	Cylindrical (exact)	Spherical	Conical
(0, 2)	3.83	3.69	3.28
(0, 3)	7.02	6.91	6.43
(1, 1)	1.75	1.49	1.02
(1, 2)	5.33	5.21	4.74
(1, 3)	8.54	8.44	7.94
(2, 1)	3.04	2.69	1.77
(2, 2)	6.71	6.59	6.07
(2, 3)	9.97	9.86	9.35

Of some interest is the effect of fill depth in a spherical container. Some results for the lowest two modes are presented graphically in Fig. 2 where the fill-depth d is defined as the maximum depth of fluid. The calculations are restricted to d not greater than the sphere radius c because of the mild-slope formulation. A solution of the shallow-water equations, valid for small fill-depth, is given by Lamb ([3], p. 291) and the calculations are in agreement with this. Though beyond the present calculation procedure, the limit for large fill-depths (d approaches $2c$) may be deduced from physical arguments. As the free-surface area tends to zero only high-frequency motion can be supported in the absence of shallow-water effects and so, in the limit, ω becomes infinite for all modes.

The result of varying the vertex angle (defined as $\pi - 2\alpha$) for a conical basin is shown in Fig. 3. In the limit as α goes to zero with the surface radius fixed the frequencies of all modes go to zero. In the opposite limit as α approaches $\pi/2$ the sides become vertical and the (exact) solution for a cylindrical container is recovered. It is perhaps worth noting that there is a range of angles around $\alpha = \pi/4$ for which there is a slower variation of frequency with α than elsewhere.

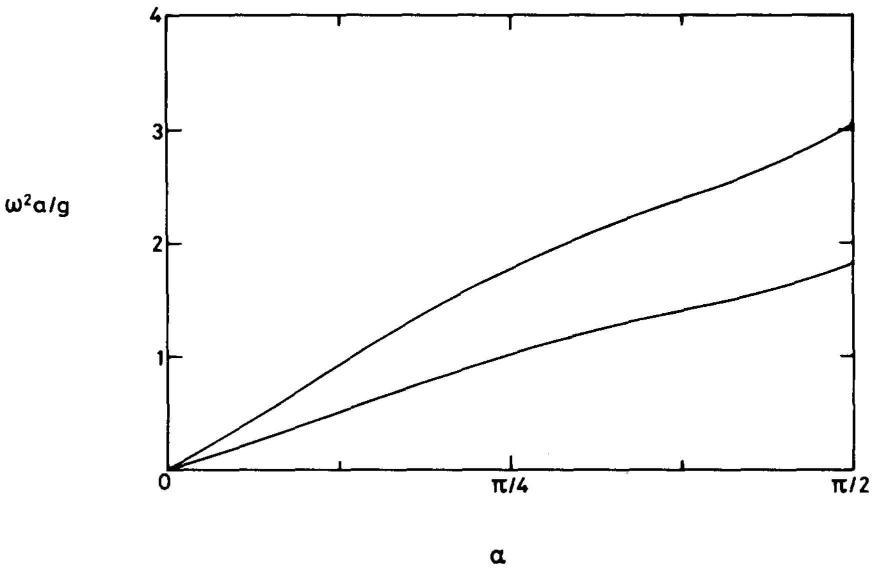

Fig. 3. The frequencies of the lowest two modes ((1, 1) and (2, 1)) for a conical container of surface radius a and vertex angle $\pi - 2\alpha$.

Appendix

From Smith and Sprinks [6],

$$q = 1 - \tfrac{1}{3}Kh + \tfrac{2}{45}(Kh)^2 + O(Kh)^3, \tag{A1}$$

$$p = \frac{Kh}{v^2}(1 - \tfrac{2}{3}Kh + O(Kh)^2). \tag{A2}$$

For a local depth profile of the form (3.6)

$$q = 1 - \tfrac{1}{3}\alpha_1 vu + \tfrac{1}{3}v(\tfrac{2}{15}\alpha_1^2 v - \alpha_2)u^2 + O(u^3), \tag{A3}$$

$$P = \frac{p}{\omega^2 a^2} = \frac{u}{v}(\alpha_1 + (\alpha_2 - \tfrac{2}{3}\alpha_1^2 v)u + O(u^2)), \tag{A4}$$

while for a depth profile (3.8)

$$q = 1 - \tfrac{1}{3}\beta v u^{1/2} + \tfrac{2}{45}\beta^2 v^2 u + O(u^{3/2}), \tag{A5}$$

$$P = \frac{p}{\omega^2 a^2} = \frac{u^{1/2}}{v}(\beta - \tfrac{2}{3}\beta^2 v u^{1/2} + O(u)). \tag{A6}$$

References

1. N.N. Moiseev and A.A. Petrov, The calculation of free oscillations in a motionless container, *Advances in Applied Mechanics* 9 (1965) 91–154.

2. D.W. Fox and J.R. Kuttler, Sloshing frequencies, *J. Applied. Maths. and Physics* (*Z.A.M.P.*) 34 (1983) 668–696.
3. H. Lamb, *Hydrodynamics*, 6th edition, Cambridge University Press (1932).
4. J.W. Miles, Surface waves in basins of variable depth, *J. Fluid Mech.* 152 (1985) 379–389.
5. C.C. Mei, *The Applied Dynamics of Ocean Surface Waves*, Wiley-Interscience (1983).
6. R. Smith and T. Sprinks, Scattering of surface waves by a conical island, *J. Fluid Mech.* 72 (1975) 373–384.
7. P. McIver and D.V. Evans, The trapping of surface waves above a submerged, horizontal cylinder, *J. Fluid Mech.* 151 (1985) 243–255.
8. C.M. Bender and S.A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers*, McGraw-Hill (1978).
9. S.R. Smith, Numerical solution of a Sturm-Liouville eigenvalue problem, Undergraduate Project Report, Brunel University (1986).
10. J.R. Kuttler and V.G. Sigillito, Sloshing of liquids in cylindrical tanks, *American Inst. of Aeronautics and Astronautics J.* 22 (1984) 309–311.
11. S.C. Chang and S.T. Wu, On the natural frequencies of standing waves in a canal of arbitrary shape, *J. Applied Math. and Physics* (*Z.A.M.P.*) 23 (1972) 881–888.
12. B. Budiansky, Sloshing of liquids in circular canals and spherical tanks, *J. Aerospace Sciences* 27 (1960) 161–173.